M381

Assignment Booklet I

Contents

		Cut-off date
2	M381 TMA 01 (Units 1 and 2 of NT and Unit 1 of ML)	10 April 2003
5	M381 TMA 02 (Units 3 and 4 of NT and Units 2 and 3 of ML)	5 June 2003

Please send your answers for each tutor-marked assignment (TMA), together with an appropriately completed assignment form (PT3), to reach your tutor on or before the cut-off date given above.

The marks allocated to each part of a question are indicated in the margin.

Although many of the questions on these assignments require numerical answers, the questions invariably carry *method marks*.

In phrasing the questions we have used the words listed below, each of which should be interpreted as indicated.

prove, **show**, **explain**	Clear reasoning and explanation for all steps are called for.
determine, **find**, **devise**, **calculate**, **compute**	An indication of the method used and all working in arriving at an answer should be given.
deduce	Clear explanation of how one result follows from another is required.
solve	Working must be shown. A numerical answer alone is not sufficient.
evaluate, **give**, **write down**	Answer suffices. No explanation need be given.
hence	No marks will be awarded for any alternative method.

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Questions 1 to 5 are on Number Theory.
Questions 6 to 8 are on Mathematical Logic.

Number Theory

Question 1 (Unit 1) – 10 marks

Use the Euclidean Algorithm to determine integers x and y such that

$$\gcd(228, 390) = 228x + 390y.$$

Write down the general solution of this linear Diophantine equation, and find the particular solution in which y takes its least possible value. [10]

Question 2 (Unit 1) – 10 marks

Use mathematical induction to prove that the following formula holds for all integers $n \geq 1$.

$$\frac{3}{1^2 \times 2^2} + \frac{5}{2^2 \times 3^2} + \frac{7}{3^2 \times 4^2} + \cdots + \frac{2n+1}{n^2 \times (n+1)^2} = 1 - \frac{1}{(n+1)^2}.$$ [10]

Question 3 (Unit 1) – 12 marks

(i) For each integer n, its square can be written uniquely as $n^2 = 10k + r$, where r is an integer with $0 \leq r \leq 9$. Determine which values of r can occur and which cannot. [4]

(ii) For which positive integer values of n is it true that $n!$ is divisible by 10? Prove your conjecture. [4]

(iii) Determine all positive integers n for which

$$1! + 2! + 3! + \cdots + n!$$

is a square.

[Hint: Consider the units digit of $1! + 2! + 3! + \cdots + n!$, using both previous parts of the question.] [4]

Question 4 (Unit 2) – 24 marks

For each of the following statements, decide whether it is true or false. If it is true, then prove it; if it is false, give a counter-example. (In this question a and b are positive integers, k and j are non-negative integers and p is prime.)

(i) If $\gcd(a, b) = p$, then $\gcd(ab, ap) = p^2$.

(ii) If $a^2 + b^2 = p^2$, then $\gcd(a, b) = 1$.

(iii) Any number of the form $12k + 5$ must have a prime divisor of the same form.

(iv) Any number of the form $12k + 7$ must have a prime divisor of one of the forms $12j + 5$ or $12j + 7$. [24]

Question 5 (Unit 2) – 9 marks

Prove that the identity

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

holds for the Fibonacci numbers F_n, for $n > 1$. [You sould not assume any properties proved about the Fibonacci numbers other than the definition that $F_1 = F_2 = 1$ and that, for $n \geq 3$, each number F_n is the sum of the previous two.] [9]

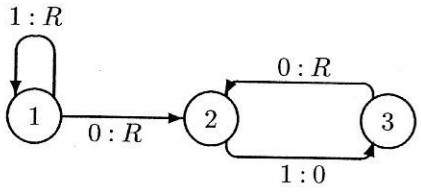
Mathematical Logic

Question 6 (Unit 1) – 10 marks

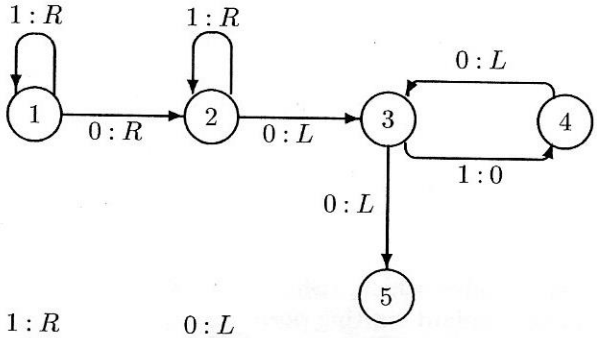
(i) We wish to design a Turing machine which, using monadic notation, inputs a pair (m, n) of positive integers in standard starting position (on an otherwise blank tape) and halts scanning the rightmost of a string of m 1s on an otherwise blank tape.

Write down which of the following Turing machines is suitable for this task. For each machine which is unsuitable, explain why it is unsuitable: this explanation can take the form of a sequence of configurations for appropriate test data.

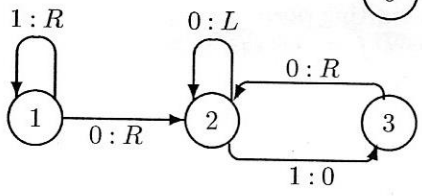
(a)



(b)



(c)

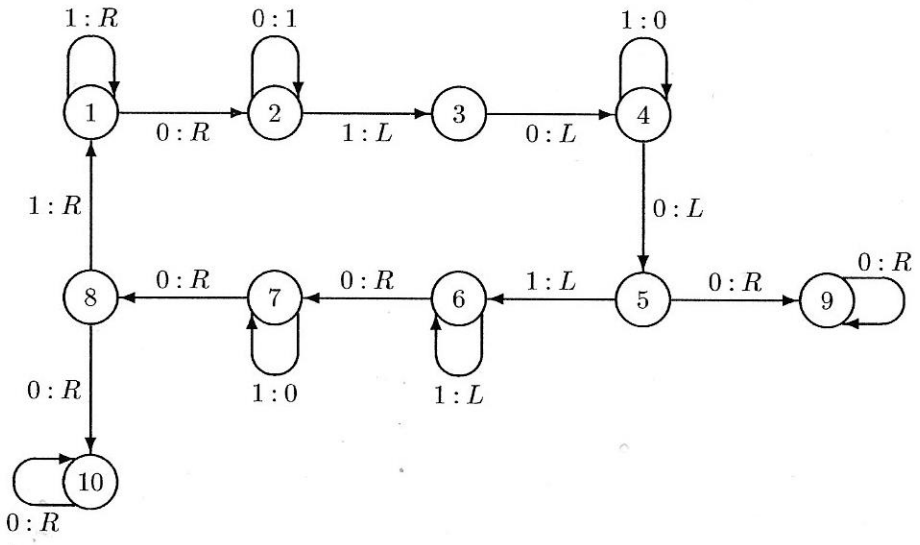


[7]

(ii) Devise and give the flow graph of a Turing machine which, if started scanning the rightmost of a string of n 2s (on an otherwise blank tape), would halt scanning a single 1 on an otherwise blank tape. [3]

Question 7 (Unit 1) – 15 marks

In this question we consider the Turing machine M with the flow graph below.



(i) Write down the machine table for M. [2]

(ii) For each of the following starting configurations of the machine M, write down the sequence of configurations for the subsequent computation.

(a) 0 1 1 0 (1) (b) 0 1 1 1 0 (1) (c) 0 1 1 1 1 0 (1) [6]

(iii) The machine M has been designed to take as input a positive integer in monadic notation and to output an integer also in monadic notation. Thus the machine computes the values of a function $f : P \longrightarrow N$.

(a) Write down the values of $f(3)$, $f(4)$, $f(5)$, $f(6)$, $f(9)$. [$2\frac{1}{2}$]

(b) What, in general, is the value of $f(n)$ for $n \in P$? Describe briefly how the machine computes $f(n)$, including an indication of each possible halting state and the circumstances under which it halts there. [$4\frac{1}{2}$]

Question 8 (Unit 1) – 10 marks

(i) Devise and give the flow graph of a Turing machine which, using monadic notation, takes as input a positive integer m in standard starting position and halts scanning the leftmost of a string of $f(m)$ 1s, where f is the function defined by

$$f(m) = \begin{cases} m, & \text{if } m \text{ is even,} \\ m+1, & \text{if } m \text{ is odd.} \end{cases}$$ [$3\frac{1}{2}$]

(ii) Devise and give the flow graph of a Turing machine which, using monadic notation, takes as input a pair (m, n) of positive integers in standard starting position and halts scanning the leftmost of a string of $g(m, n)$ 1s, where g is the function defined by

$$g(m,n) = \begin{cases} m, & \text{if } m \text{ is even,} \\ n, & \text{if } m \text{ is odd.} \end{cases}$$ [$4\frac{1}{2}$]

(iii) Write down the sequences of configurations for the computations of your machines in parts (i) and (ii) which evaluate the following values of the functions f and g.

(a) $f(2)$ (b) $f(3)$ (c) $g(1,2)$ (d) $g(2,2)$ [2]

Questions 1 to 4 are on Number Theory.
Questions 5 to 9 are on Mathematical Logic.

Number Theory

Question 1 (Unit 3) – 10 marks

Determine the least positive integer which simultaneously satisfies all three of the following congruences:

$$x \equiv 1 \pmod 3;$$
$$x \equiv 1 \pmod 7;$$
$$3(x+2) \equiv 5 \pmod{19}.$$

[10]

Question 2 (Unit 3) – 15 marks

The parts of this question lead to a solution of the following.

Problem: Which positive integers (if any) have a square which ends in four equal digits (i.e. ends in 0000, or 1111, or 2222, etc.)?

(i) Write down the digits that cannot occur as the final digit of a square. [1]

(ii) Using the facts that, for any integer n, $n^2 \equiv 0$ or $1 \pmod 4$ and that any integer N is congruent modulo 4 to the number made up of the last two digits of N, show that a square cannot end in the pair of digits 11. Which other pairs of equal digits can be eliminated similarly? [4]

(iii) Determine all integers r, $0 \leq r < 16$, to which a square may be congruent modulo 16. [3]

(iv) Prove that any integer terminating in the four equal digits 4444 is congruent modulo 16 to 12. Can such a number be a square? Explain your answer. [4]

(v) Answer the problem stated in the preamble. [3]

Question 3 (Unit 4) – 12 marks

All parts of this question involve FLT.

(i) Find the least positive residue of $7^{50} \pmod{43}$. [3]

(ii) Solve $8x \equiv 1 \pmod{43}$. Explain why the solution of this congruence is equal to the least positive residue of $8^{41} \pmod{43}$. Write down a linear congruence (modulo 43) whose solution is congruent to $8^{40} \pmod{43}$, and hence determine the least positive residue of $8^{40} \pmod{43}$. [6]

(iii) Prove that $a^{25} \equiv a \pmod{195}$ for every integer a. [3]

Question 4 (Unit 4) – 13 marks

All the proofs in this question involve application of Wilson's Theorem or its converse.

(i) Prove that, for any odd integer $n > 2$, n and $n+2$ are twin primes if and only if

$$4[(n-1)!+1]+n \equiv 0 \pmod{n(n+2)}.$$

[Hint: You will need to consider this congruence to moduli n and $n+2$ separately.] [8]

(ii) Calculate the least positive residue of

$$(2p-1)! \pmod{p^2}$$

for $p = 2$, 3 and 5. Conjecture a result about this least positive residue for a general prime p, and prove your conjecture.

[Hint: Show that

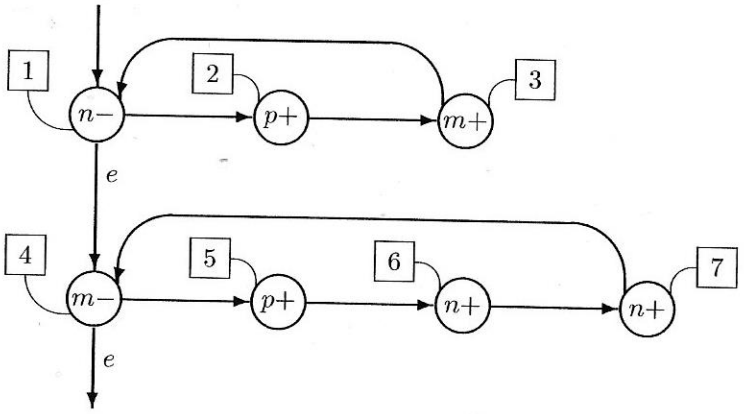
$$(p+1)(p+2)(p+3)\ldots(2p-1) \equiv -1 \pmod{p},$$

and hence that this product is equal to $kp-1$, for some integer k.] [5]

Mathematical Logic

Question 5 (Unit 2) – 10 marks

This question concerns the abacus machine whose flow chart is shown below.



(i) Write down the trace table for the computation of this machine when initially the contents of the registers are:

$$[m] = 1; \quad [n] = 1; \quad [p] = 2.$$ [4]

(ii) Suppose that m, n, p initially contain respectively the first, second and third arguments of a function f and that the value of f is given by the content of register p when the computation halts.

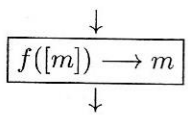
(a) Write down a formula which describes the rule of f. [2]

(b) How are the final contents of registers m and n related to the initial contents of the three registers? [4]

Question 6 (Unit 2) – 10 marks

In both parts of this question you may use extra registers besides the ones mentioned, if you so wish.

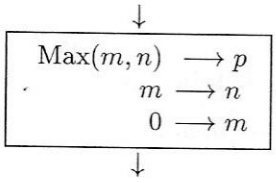
(i) Give in full the flow chart of an abacus machine program which has the effect shown in the following diagram.



where $f(x) = \begin{cases} x/4, & \text{if } x \text{ is exactly divisible by 4,} \\ x/2, & \text{if the remainder when } x \text{ is divided by 4 is 2,} \\ 0, & \text{otherwise.} \end{cases}$

[5]

(ii) Give in full the flow chart of an abacus machine program which has the effect shown in the following diagram.



where $\text{Max}(x,y) = \begin{cases} y, & \text{if } x \leq y, \\ x, & \text{if } y < x. \end{cases}$

Assume that $[p] = 0$ initially. [5]

Question 7 (Unit 3) – 12 marks

(i) Let h be the function

$$\text{Pr}[\text{Cn}[s, s], \text{Cn}[\exp, \text{Cn}[s, \text{id}_3^3], \text{Cn}[s, \text{id}_2^3]]],$$

where exp is the product function defined by $\exp(x, y) = x^y$. Determine the values of $h(4, 0)$ and $h(3, 2)$, writing down the stages of your calculation. [6]

(ii) Consider the function $h : N \times N \times N \longrightarrow N$ defined by

$$h(x_1, x_2, x_3) = x_2 \cdot (x_1 + x_3).$$

Using the operations of composition and primitive recursion in terms of the *basic* functions (i.e. the zero, successor and identity functions) and, if you wish, the sum function, give a formal definition which shows h to be primitive recursive. [6]

Question 8 (Unit 3) – 12 marks

In this question you may use any of the primitive recursive functions or results about such functions discussed in Unit 3. You may present your arguments using either formal or informal definitions of the functions involved. Please give your tutor a reference to any result you use.

(i) Show that the function f defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is even,} \\ 0, & \text{if } x \text{ is odd,} \end{cases}$$

is primitive recursive. [3]

(ii) Using the function of part (i), or otherwise, show that the function g defined by

$$g(x) = \begin{cases} x^2, & \text{if } x \text{ is even,} \\ 0, & \text{if } x \text{ is odd,} \end{cases}$$

is primitive recursive. [3]

(iii) Using the result of part (ii), or otherwise, show that the function h defined by

$$h(x) = \text{the sum of the squares of all the even numbers less than } x$$

is primitive recursive. [3]

(iv) Show that the function k defined by

$$k(x) = \text{the sum of the squares of all the even numbers less than or equal to } x$$

is primitive recursive. [3]

Question 9 (Unit 3) – 6 marks

The function f is defined by

$$f(x, y) = (x \dot{-} y^3) + (y^3 \dot{-} x).$$

(i) Compute the values of $\mathrm{Mn}[f](1)$ and $\mathrm{Mn}[f](27)$. [2]

(ii) Explain why $\mathrm{Mn}[f](3)$ is not defined. [1]

(iii) For which natural numbers x is it the case that $\mathrm{Mn}[f](x)$ is defined? Give a brief justification of your answer. [3]

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M381

Assignment Booklet II

Contents **Cut-off date**

2	TMA M381 03 (Units 5 and 6 of NT and Units 4 and 5 of ML)	7 August 2003
6	TMA M381 04 (Units 7 and 8 of NT and Units 6 and 7 of ML)	25 September 2003

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Number Theory

Question 1 (Unit 5) – 12 marks

(i) Determine the smallest positive integer which has exactly 45 distinct positive divisors including itself and 1. [3]

(ii) Prove that for all positive integers n, $\sigma(2n) > 2\sigma(n)$ (where $\sigma(n)$ denotes the sum of the divisors of n). [3]

(iii) An integer n with the property $\sigma(n) > 2n$ is said to be *abundant*.

(a) Determine which, if either, of 195 and 196 is abundant. [2]

(b) Suppose that $n = 2^r p$, where $r \geq 1$ and p is an odd prime. Show that for every odd prime p we can find values of r for which n is abundant. List all the abundant numbers of this form which are less than 100. [4]

Question 2 (Unit 5) – 13 marks

A function F is defined for all positive integers by

$$F(n) = \begin{cases} \text{the number of integers } a \text{ in} \\ \text{the range } 1 \leq a \leq n \text{ for which} \\ \gcd(a, n) = \gcd(a+1, n) = 1. \end{cases}$$

So, for example, $F(15) = 3$ since

$$\text{for } a = 1: \quad \gcd(1, 15) = \gcd(2, 15) = 1;$$
$$\text{for } a = 7: \quad \gcd(7, 15) = \gcd(8, 15) = 1;$$
$$\text{for } a = 13: \quad \gcd(13, 15) = \gcd(14, 15) = 1;$$

whilst for the other 12 values of a in the range $1 \leq a \leq n$, either $\gcd(a, 15) > 1$ or $\gcd(a+1, 15) > 1$.

(i) Write down the values of $F(7)$, $F(8)$, $F(9)$ and $F(12)$. [2]

(ii) Show that $F(n) = 0$ if and only if n is even. [2]

(iii) Determine the value of $F(p)$, where p is any prime. [2]

(iv) Determine the value of $F(p^r)$, where p is prime and $r \geq 2$. [5]

(v) Given that F is a multiplicative function, determine the value of $F(675)$ and complete the following formula by supplying the entry missing from the bracket:

$$F(n) = n \prod_{p|n} (\quad). \qquad [2]$$

Question 3 (Unit 6) – 14 marks

(i) Determine whether or not the quadratic congruence

$$5x^2 + 7x + 3 \equiv 0 \pmod{17}$$

has solutions:

(a) by using Euler's Criterion; [2]

(b) by using Gauss' Lemma. [2]

(ii) Using the *LQR* in conjunction with Theorems 2.1 and 3.2 of *Unit 6*, determine the values of the Legendre symbols

$$(-8/47),\ (227/31) \text{ and } (227/307).$$

[Note that each of 31, 47, 227 and 307 is prime.] [6]

(iii) Determine all primes $p > 3$ for which $(-6/p) = 1$. Give your answer in the form of a condition on p modulo n for an appropriate value of n. [4]

Question 4 (Unit 6) – 11 marks

In Theorem 3.5 of *Unit 6*, we proved that there are infinitely many primes of the form $8k + 7$, where $k \geq 0$ is an integer. In this question we ask you to prove that same result, but by a slightly different method.

Suppose that there are only finitely many primes of the form $8k + 7$, say p_1, p_2, $p_3, \ldots, p_n$. Consider the number

$$N = (p_1 p_2 p_3 \ldots p_n)^2 - 2.$$

(i) Show that $N \equiv 7 \pmod{8}$. [2]

(ii) Show that if p is a prime divisor of N, then $(2/p) = 1$. [2]

(iii) Deduce that one of the primes $p_1, p_2, p_3, \ldots, p_n$ must be a divisor of N, and thereby reach a contradiction. [7]

Mathematical Logic

In your answers to Questions 5 and 6 you are advised to show functions to be primitive recursive by use of informal, rather than formal, definitions. You may, of course, make use of results derived in the units, or mentioned in the Handbook.

Question 5 (Unit 4) – 10 marks

Show that the function f of two arguments defined by

$$f(x,y) = \begin{cases} xy^3, & \text{if } x + y = 80 \text{ and } x > 21, \\ xy, & \text{if } x + 3y \geq 200, \\ 7, & \text{otherwise,} \end{cases}$$

is primitive recursive. [10]

Question 6 (Unit 4) – 10 marks

(i) Show that the function c defined by

$$c(x,y) = \begin{cases} 1, & \text{if } x \text{ is divisible by } y^2, \\ 0, & \text{otherwise,} \end{cases}$$

is primitive recursive. [3]

(ii) Show that the function h defined for positive integers by

$$h(x) = \begin{cases} 1, & \text{if } x \text{ is square-free, i.e. is not divisible by the square} \\ & \text{of any integer except 1,} \\ 0, & \text{otherwise,} \end{cases}$$

is primitive recursive. (Note that, for example, $h(1) = 1$, $h(2) = 1$, $h(4) = 0$, $h(6) = 1$, $h(18) = 0$.) [7]

[*Hint*: One method of approach is to consider how the number of ys less than or equal to x for which x is divisible by y^2 is related to the problem of deciding whether x is square-free.]

Question 7 (Unit 4) – 5 marks

(i) Draw the flow graph of the Turing machine M given by the following tables of values of $a(x,y)$ and $q(x,y)$.

$a(x,y)$	y	
x	0	1
1	0	2
2	3	2
3	1	1
4	0	1
5	3	2

$a(x,y) = y$ otherwise

$q(x,y)$	y	
x	0	1
1	0	3
2	1	2
3	5	0
4	0	0
5	2	3

$q(x,y) = 0$ otherwise

[2]

(ii) At a certain stage in M's computation on a certain input, the left number and right number describing M's tape configuration are 71 and 38 respectively, and M is in state 5.

(a) Produce a diagram of M's tape at this stage.

(b) Calculate the left and right numbers after the next step in M's computation. [3]

Question 8 (Unit 5) – 10 marks

(i) For each of the following strings of symbols, state whether or not the string is a formula. (Simply answer YES or NO.)

(a) $\forall x\, \exists y\, ((x \cdot y \cdot z) + z) = (y + x)$

(b) $\exists y\, (\forall x\, x = y' \vee \forall y\, (\mathbf{0}' + y) = t)$

(c) $\forall y\, \exists t\, ((t + y)' \cdot z)$ [3]

(ii) For each string in part (i) which is *not* a formula, explain briefly why it is not a formula. [2]

(iii) By choosing suitable subformulas, show that the formula

$$((-\forall x\, x' = y \rightarrow -x' = y) \rightarrow (-(x' = y \,\&\, \exists x\, (x' = y \leftrightarrow \forall x\, x' = y)) \vee \forall x\, x' = y))$$

has truth value 1 under all interpretations of the symbols used in the formula. [5]

Question 9 (Unit 5) – 15 marks

(i) Parts (a), (b) and (c) below are concerned with the following (contorted but correct) formal proof.

(1)	$-(\phi \,\&\, -\theta)$	Ass
(2)	$(-\theta \to -\phi)$	Taut, (1)
(3)	$(-\psi \vee \phi)$	Ass
(4)	$(-(\phi \,\&\, -\theta) \to (-\theta \to -\phi))$	CP, (2)
(5)	$((-\psi \vee \phi) \to (\psi \to \theta))$	Ass
(6)	$(\psi \to \theta)$	Taut, (2), (3)
(7)	$((-\psi \vee \phi) \to (\psi \to \theta))$	Taut, (5)
(8)	$((-\psi \vee \phi) \to (\psi \to \theta))$	CP, (6)

(a) The assumption numbers are missing from the above proof. Write down what they should be for each line of the proof. [4]

(b) State which formulas have been used where the Tautology Rule has been applied (on lines (2), (6) and (7)). [3]

(c) Show that the formula in your answer to part (b) which is used to obtain line (6) is indeed a tautology. [2]

(ii) Write down a formal proof of the formula

$$((\phi \longleftrightarrow -\psi) \to (\psi \to -\phi))$$

depending on no assumptions. State which tautologies you have used whenever the Tautology Rule has been applied in your proof. [6]

Questions 1 to 5 are on Number Theory.
Questions 6 to 9 are on Mathematical Logic.

Number Theory

Question 1 (Unit 7) – 12 marks

(i) Determine the simple continued fraction of $71/19$ and hence obtain the general solution of the linear Diophantine equation

$$71x - 19y = 2.$$ [4]

(ii) Determine the continued fraction of

$$\frac{3+\sqrt{14}}{5}.$$ [4]

(iii) Write down the convergents $C_1, C_2, C_3, \ldots, C_7$ of the infinite continued fraction $[1,1,1,3,1,5,1,7,\ldots]$. Let β be the irrational number with this continued fraction. Without using a calculator, determine the least value of n for which $|\beta - C_n| < 0.001$, justifying your answer. [4]

Question 2 (Unit 7) – 13 marks

Consider the two continued fractions

$$x = [a, 1, b_1, b_2, b_3, b_4, \ldots] \quad \text{and} \quad y = [a, 1 + b_1, b_2, b_3, b_4, \ldots].$$

(i) Letting $\beta = [b_1, b_2, b_3, b_4, \ldots]$, establish the formulae

$$(\beta+1)(x-a) = \beta \quad \text{and} \quad (\beta+1)(y-a) = 1.$$ [4]

(ii) By eliminating β from the formulas in part (i), show that

$$x + y = 2a + 1.$$ [1]

(iii) From the formula in part (ii), noting that $2a+1$ is an integer, write down the simple infinite continued fractions for $-x$ and $-y$. [2]

(iv) Let

$$x_1 = [1, 1, 2, 2, 3, 3, \ldots]$$

and

$$x_2 = [1, 2, 3, 4, 5, 6, \ldots].$$

Write down the continued fractions of $-x_1$ and $-x_2$. [2]

(v) Write down the continued fraction of $-[\langle a, b\rangle]$, where a and b are any positive integers. [4]

Question 3 (Units 7 and 8) – 12 marks

(i) Show that the continued fraction of $\sqrt{20}$ is $[4, \langle 2, 8\rangle]$. [6]

(ii) Use the continued fraction in part (i) to find three positive solutions of the Diophantine equation $x^2 - 20y^2 = 1$. [4]

(iii) From the solutions in part (ii), deduce three positive solutions of the Diophantine equation $x^2 - 8xy - 4y^2 = 1$. [2]

Question 4 (Unit 8) – 6 marks

One of the numbers 1150 and 1305 can be written as a sum of two squares, and the other cannot. Explain why one cannot be so written, and write the other as a sum of two positive squares in *two* different ways. (Your working should be shown as in Example 4.1 of *Unit 8*.)

[6]

Question 5 (Unit 8) – 7 marks

The objective of this question is to use the method of infinite descent to show that $\sqrt{11}$ *is irrational.*

Suppose that $\sqrt{11} = \dfrac{m}{n}$, where m and n are positive integers. Show that

$$\sqrt{11} = \frac{11n - 3m}{m - 3n},$$

and explain how the method of infinite descent can be used to deduce from this that $\sqrt{11}$ is irrational.

[7]

Mathematical Logic

Question 6 (Unit 6) – 5 marks

(i) For each of the following formulas, determine which occurrences of variables are free and which are bound.

(a) $(\forall t\,(x + t) = y \vee \exists x\,(y \cdot t) = x)$

(b) $\exists y\,\exists t\,(\exists x\,x' = y \,\&\, t' = y)$

(c) $\forall t\,\exists y\,(\forall x\,x = t \to y = x')$

[$2\frac{1}{2}$]

(ii) For each of the formulas in (a), (b) and (c) above, write down the result of substituting the term $(y + x)$ for each free occurrence of x in the formula. In which of the formulas may this term be freely substituted for x?

[$2\frac{1}{2}$]

Question 7 (Unit 6) – 20 marks

Write down formal proofs to establish the following results.

(i) $\forall x\,(\phi \,\&\, \psi) \vdash \forall x\,(\phi \to \psi)$, where ϕ, ψ are any formulas. [4]

(ii) $\exists x\,{-\psi} \vdash (\forall x\,(\phi \vee \psi) \to \phi)$, where the variable x does not occur free in ϕ, indicating at which step(s) of your proof this condition on ϕ is required. [6]

(iii) $\forall x\,(\phi \,\&\, {-\theta}) \vdash (\exists x\,(\theta \longleftrightarrow {-\psi}) \to {-\forall x\,(\psi \to {-\phi})})$ [6]

(iv) $x = z \vdash ((z + y) \cdot x) = ((x + y) \cdot z)$ [4]

Question 8 (Unit 7) – 15 marks

Write down formal proofs to show the following.

(i) $\vdash_Q (\mathbf{3} + \mathbf{1}) \neq \mathbf{3}$,
that is, $\vdash_Q {-(\mathbf{0}''' + \mathbf{0}')} = \mathbf{0}'''$ [7]

(ii) $\vdash_Q \exists y\,(y' \cdot y) = (y + \mathbf{0}')$ [8]

Question 9 (Unit 7) – 10 marks

One of the following sentences is a theorem of Q, and one is not. Give a formal proof of the sentence that is a theorem of Q, and explain why the other sentence is not a theorem of Q. (You may assume that the axioms of Q are true in both the interpretations N^* and N^{**} given on pages 5 and 6 of the *Logic Handbook*.)

(i) $\forall x \forall y ((x \cdot y') \cdot \mathbf{0}) = (x \cdot (y' \cdot \mathbf{0}))$

(ii) $\forall x \forall y ((x \cdot \mathbf{0}) \cdot y') = (x \cdot (\mathbf{0} \cdot y'))$ [10]

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